Distributed convergence to Nash equilibria in two-network zero-sum games

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Joint work with **Bahman Gharesifard**

My connections to Mark

- Met him at CSL@UIUC as visiting grad student, later as postdoc
- Had read by then several of his papers on passivity, haptics, and teleoperation
- Almost had him as my dean at UC Santa Cruz (but I left and he turned down the offer)
- Had read by then several of his papers on multi-agent systems
- Worked with him in organizing committee of CDC10 (Mark was General Chair)

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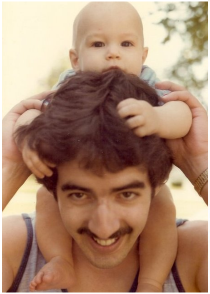
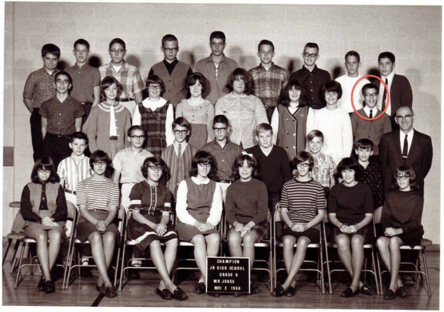
- And of course, **facebook!**

Fun photos I've found on facebook

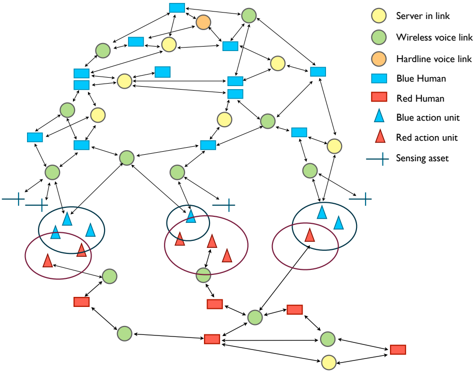




Life as Dean of UTD is good!



Networked strategic scenarios



information is distributed across multiple layers

partial, evolving, dynamic interactions

agents cooperating and competing with each other

Individual **agents**, not networks, are **decision makers**

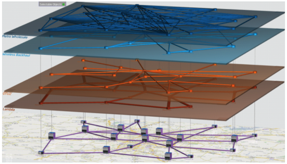
Literature on games played on networks

In Economics, **network games:** equilibria characterization when agents have incomplete information (e.g., known degree but unknown neighbor identities)

In Computer Science, **graphical games:** algorithms (and their complexity) to compute equilibria in two-action complete information games on networks

In Controls, current interest on

- coordination in adversarial teams
- multi-layer scenarios with interacting agents
- distributed learning with partial agent knowledge of global information



What are we after

Emphasis not on equilibria, but on how to get there

Objective:

coordination algorithms to help agents decide how to play the game

under partial information, local interactions

Characterization of **algorithm features** regarding

- performance gap between distributed and centralized setups
- robustness to changing interactions, noise, message dropping
- preservation of private information

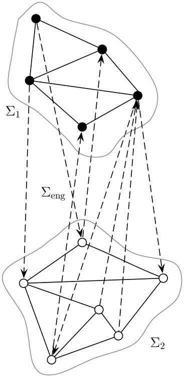
Outline

1. Problem statement
   - Two-network zero-sum game
   - Primer on graph theory

2. Distributed convex optimization
   - One-network problem
   - Distributed dynamics and convergence

3. Distributed convergence to Nash equilibria
   - Reformulation of the two-network zero-sum game
   - Distributed dynamics and convergence
   - Dynamic interaction topologies and robustness to link failures

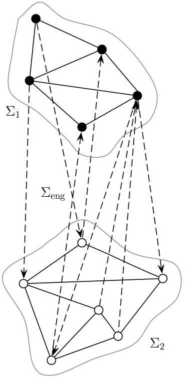
Two-network zero-sum game



Each **player** is a **network** of cooperating agents

- $x_1 \in \mathsf{X}_1 \subset \mathbb{R}^{d_1}$ state of Σ_1, $x_2 \in \mathsf{X}_2 \subset \mathbb{R}^{d_2}$ state of Σ_2

Two-network zero-sum game



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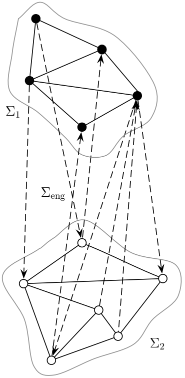

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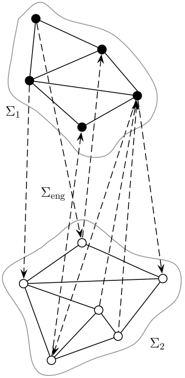

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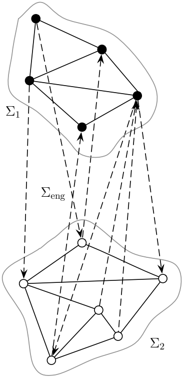
- **payoff function**

$$U(x_1, x_2) = \sum_{i=1}^{n_1} f_1^i(x_1, x_2) = \sum_{j=1}^{n_2} f_2^j(x_1, x_2)$$

f_1^i available to i in Σ_1, f_2^j available to j in Σ_2

Two-network zero-sum game



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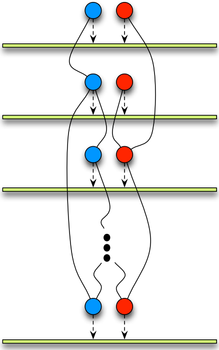
f_1^i available to i in Σ_1, f_2^j available to j in Σ_2

Σ_1 wishes to **maximize** U, Σ_2 wishes to **minimize** U

Scenario with n communication channels



- **capacity** of ith channel is proportional to

$$\log(1 + \beta p_i / (\sigma_i + \eta_i))$$

with signal power p_i, noise power η_i, receiver noise σ_i

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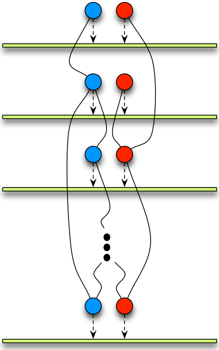

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$$\sum_{i=1}^{n} p_i = P \qquad \sum_{i=1}^{n} \eta_i = C$$

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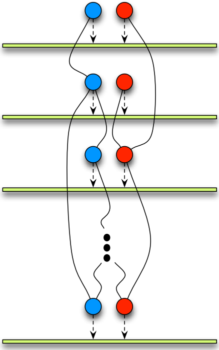

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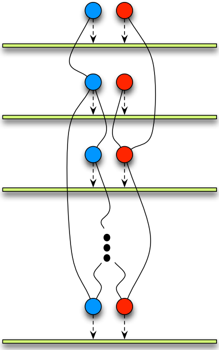
- Blue team selects
  - m_1 channels with signal power x_1,
  - $n - 1 - m_1$ with x_2,
  - one channel with $P - m_1 x_1 - (n - 1 - m_1)x_2$

Scenario with n communication channels



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  - $n - 1 - m_1$ with x_2,
  - one channel with $P - m_1 x_1 - (n - 1 - m_1) x_2$
- Red team similarly

Sample scenario - cont'd

Scenario fits **two-network zero-sum game paradigm**

Capacity of ith channel

$$f^i(x,y) = \log\left(1 + \frac{\beta x_a}{\sigma_i + y_b}\right) \quad \text{(for some } a, b \in \{1,2\}\text{)}$$

Capacity of nth channel

$$f^n(x,y) = \log\left(1 + \frac{\beta(P - m_1 x_1 - (n-1-m_1)x_2)}{\sigma_n + C - m_2 y_1 - (n-1-m_2)y_2}\right)$$

Blue/red teams seek to maximize/minimize total capacity

From the agent's viewpoint

Network only knows objective function collectively, not at agent level

$$x_1^i \in \mathsf{X}_1 \text{ is estimate of agent } i \text{ about state } x_1 \text{ of } \Sigma_1$$
$$x_2^j \in \mathsf{X}_2 \text{ is estimate of agent } j \text{ about state } x_2 \text{ of } \Sigma_2$$

Collective estimates:

$$\boldsymbol{x}_1 = (x_1^1, \dots, x_1^{n_1}) \in (\mathbb{R}^{d_1})^{n_1} \quad \boldsymbol{x}_2 = (x_2^1, \dots, x_2^{n_2}) \in (\mathbb{R}^{d_2})^{n_2}$$

Objective: through distributed interactions,

- agents on Σ_1 agree on state $\boldsymbol{x}_1^* = \mathbf{1}_{d_1} \otimes x_1^* = (x_1^*, \dots, x_1^*)$
- agents on Σ_2 agree on state $\boldsymbol{x}_2^* = \mathbf{1}_{d_2} \otimes x_2^* = (x_2^*, \dots, x_2^*)$
- (x_1^*, x_2^*) is Nash equilibrium of 2-network zero-sum game

$\otimes$ is Kronecker product

Graph-theoretic notions

Network topology modeled via **undirected graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

- $\mathcal{V}$ is set of agent identities
- $\mathcal{E}$ is set of edges between agents – information sharing

Relevant **matrices** and their properties

- $\mathcal{A}$ is **adjacency** matrix (who interacts with whom)
- $L = \text{diag}(\mathcal{A}\mathbf{1}_n) - \mathcal{A}$ is **Laplacian** matrix
- L positive semidefinite
- $L\mathbf{1}_n = 0$ (0 is an eigenvalue of L)
- $\mathcal{G}$ is connected if and only if $\text{rank}(L(\mathcal{G})) = n - 1$

One-network problem

Simpler setup with only one network: objective is to minimize

$$f(x) = \sum_{i=1}^{n} f^i(x)$$

Reformulation for network of agents:

- agent i has own estimate x^i, so $\boldsymbol{x} = (x^1, \dots, x^n)$
- all agents should agree on minimizer, $\boldsymbol{x} = \mathbf{1}_n \otimes x$

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Problem reformulated on $\mathbb{R}^{nd}$,

$$\begin{aligned} &\text{minimize} \quad \tilde{f}(\boldsymbol{x}) = \sum_{i=1}^{n} f^i(x^i) \\ &\text{subject to} \;\; \mathbf{L}\boldsymbol{x} = \mathbf{0}_{nd} \end{aligned}$$

Solutions as saddle points

For $\mathcal{G}$ connected, $\{f^i\}_{i=1}^n$ differentiable and convex, let $F : \mathbb{R}^{nd} \times \mathbb{R}^{nd} \to \mathbb{R}$

$$F(\boldsymbol{x}, \boldsymbol{z}) = \tilde{f}(\boldsymbol{x}) + \boldsymbol{x}^T \mathbf{L} \boldsymbol{z} + \frac{1}{2} \boldsymbol{x}^T \mathbf{L} \boldsymbol{x}$$

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Proposition

F is differentiable, convex in its first argument and linear in its second,

1. *if $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ is saddle point of F, then $\boldsymbol{x}^*$ is a solution*
2. *if $\boldsymbol{x}^*$ is a solution, then there exists $\boldsymbol{z}^*$ with $\mathbf{L}\boldsymbol{z}^* = -\nabla \tilde{f}(\boldsymbol{x}^*)$ such that $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ is saddle point of F*

Distributed solution to optimization problem

Saddle-point dynamics of F is distributed!

From **network** viewpoint,

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= -\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{z} - \nabla\tilde{f}(\boldsymbol{x}) \\
\dot{\boldsymbol{z}} &= \mathbf{L}\boldsymbol{x}
\end{aligned}
$$

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From **agent** viewpoint,

$$\dot{x}^i = -\sum_{k\in\mathcal{N}^i}(x^i - x^k) - \sum_{k\in\mathcal{N}^i}(z^i - z^k) - \nabla f^i(x^i)$$

$$\dot{z}^i = \sum_{k\in\mathcal{N}^i}(x^i - x^k)$$

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Theorem

For $\mathcal{G}$ connected and $\{f^i\}_{i=1}^n$ differentiable and convex,

the projection onto first component of trajectories asymptotically converges to solution set

Can be extended to locally Lipschitz and convex functions (not differentiable)

If solution set is finite, then convergence to **a** solution is guaranteed

Lifting the zero-sum game

Recall **objective** is

- agents on Σ_1 agree on state $\boldsymbol{x}_1^* = \mathbf{1}_{d_1} \otimes x_1^*$ $\iff$ $\mathbf{L}_1 \boldsymbol{x}_1 = \mathbf{0}_{n_1 d_1}$
- agents on Σ_2 agree on state $\boldsymbol{x}_2^* = \mathbf{1}_{d_2} \otimes x_2^*$ $\iff$ $\mathbf{L}_2 \boldsymbol{x}_2 = \mathbf{0}_{n_2 d_2}$
- (x_1^*, x_2^*) is Nash equilibrium of 2-network zero-sum game

Lifting the zero-sum game

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Evaluation of objects like $f_1^i(x_1, x_2)$ requires

- estimate of own network's state x_1^i
- estimate of other network's state info from neighbors in Σ_{eng}

Lifting the zero-sum game

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Each agent in Σ_1 has $\mathfrak{f}_1^i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2 |\mathcal{N}^{\text{in}}_{\Sigma_{\text{eng}}}(v_i)|} \to \mathbb{R}$ concave-convex such that

$$\tilde{f}_1^i(x_1, x_2, \ldots, x_2) = f_1^i(x_1, x_2)$$

For convenience, $\tilde{f}_1^i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2 n_2} \to \mathbb{R}$, $\tilde{f}_1^i(x_1, \boldsymbol{x}_2) = \mathfrak{f}_1^i(x_1, \pi_1^i(\boldsymbol{x}_2))$

Similar construction for agents in Σ_2

($\pi_1^i(\boldsymbol{x}_2)$ are values received by from neighbors in Σ_{eng})

Characterization of Nash equilibria via saddle property

For Σ_1, Σ_2 connected, let

$$F_1(\boldsymbol{x}_1, \boldsymbol{z}_1, \boldsymbol{x}_2) = -\tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{x}_1^T \mathbf{L}_1 \boldsymbol{z}_1 + \frac{1}{2}\boldsymbol{x}_1^T \mathbf{L}_1 \boldsymbol{x}_1$$

$$F_2(\boldsymbol{x}_2, \boldsymbol{z}_2, \boldsymbol{x}_1) = \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{x}_2^T \mathbf{L}_2 \boldsymbol{z}_2 + \frac{1}{2}\boldsymbol{x}_2^T \mathbf{L}_2 \boldsymbol{x}_2$$

$(\boldsymbol{x}_1^*, \boldsymbol{z}_1^*, \boldsymbol{x}_2^*, \boldsymbol{z}_2^*)$ satisfies **(F_1, F_2)-saddle property** if

- $(\boldsymbol{x}_1^*, \boldsymbol{z}_1^*)$ saddle of $(\boldsymbol{x}_1, \boldsymbol{z}_1) \mapsto F_1(\boldsymbol{x}_1, \boldsymbol{z}_1, \boldsymbol{x}_2^*)$
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Proposition

F_1 and F_2 convex in first argument, linear in second, and concave in third,

1. *If $(\boldsymbol{x}_1^*, \boldsymbol{z}_1^*, \boldsymbol{x}_2^*, \boldsymbol{z}_2^*)$ satisfies (F_1, F_2)-saddle property, then $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)$ is Nash equilibrium of $\mathbf{G}_{\text{adv-net}}$*
2. *if $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)$ is Nash equilibrium of $\mathbf{G}_{\text{adv-net}}$, then there exists $\boldsymbol{z}_1^*, \boldsymbol{z}_2^*$ such that $(\boldsymbol{x}_1^*, \boldsymbol{z}_1^*, \boldsymbol{x}_2^*, \boldsymbol{z}_2^*)$ satisfies saddle property for (F_1, F_2)*

Distributed solution to 2-network zero-sum game

'Saddle-point dynamics for (F_1, F_2)' is distributed!

From network viewpoint,

$$\begin{aligned}
\dot{\boldsymbol{x}}_1 &= -\mathbf{L}_1 \boldsymbol{x}_1 - \mathbf{L}_1 \boldsymbol{z}_1 + \nabla_{\boldsymbol{x}_1} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\dot{\boldsymbol{z}}_1 &= \mathbf{L}_1 \boldsymbol{x}_1 \\
\dot{\boldsymbol{x}}_2 &= -\mathbf{L}_2 \boldsymbol{x}_2 - \mathbf{L}_2 \boldsymbol{z}_2 - \nabla_{\boldsymbol{x}_2} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\dot{\boldsymbol{z}}_2 &= \mathbf{L}_2 \boldsymbol{x}_2
\end{aligned}$$

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From agent viewpoint,

$$\dot{x}_1^i = \sum_{k\in\mathcal{N}_1^i}(x_1^k - x_1^i) + \sum_{k\in\mathcal{N}_1^i}(z_1^k - z_1^i) + \nabla_{x_1^i}\tilde{f}_1^i(x_1^i, \boldsymbol{x}_2)$$

$$\dot{z}_1^i = \sum_{k\in\mathcal{N}_1^i}(x_1^i - x_1^k)$$

$$\dot{x}_2^j = \sum_{l\in\mathcal{N}_2^j}(x_2^l - x_2^j) + \sum_{l\in\mathcal{N}_2^j}(z_2^l - z_2^j) - \nabla_{x_2^j}\tilde{f}_2^j(\boldsymbol{x}_1, x_2^j)$$

$$\dot{z}_2^j = \sum_{l\in\mathcal{N}_2^j}(x_2^j - x_2^l)$$

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\dot{\boldsymbol{x}}_2 &= -\mathbf{L}_2 \boldsymbol{x}_2 - \mathbf{L}_2 \boldsymbol{z}_2 - \nabla_{\boldsymbol{x}_2} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\dot{\boldsymbol{z}}_2 &= \mathbf{L}_2 \boldsymbol{x}_2
\end{aligned}$$

Theorem (Hierarchy of saddle-point problems)

For zero-sum game, with Σ_1 and Σ_2 connected, $U : \mathsf{X}_1 \times \mathsf{X}_2 \to \mathbb{R}$ differentiable and strictly concave-convex that admits lift to $\tilde{U}$,

> *projection onto the first and third components of trajectories asymptotically converge to agreement on the Nash equilibrium*

Can also be extended to locally Lipschitz (not differentiable) scenario

Proof summary and consequences

Proof uses careful combination of

- stability analysis (Lyapunov function + LaSalle Invariance Principle)
- convexity analysis (first-order condition of convexity, interplay F_1 and F_2)
- nonsmooth analysis (in the locally Lipschitz case)

Interestingly, Lyapunov function does not depend on particular graphs

$$\begin{aligned} V(\boldsymbol{x}_1, \boldsymbol{z}_1, \boldsymbol{x}_2, \boldsymbol{z}_2) = {} & \frac{1}{2}(\boldsymbol{x}_1 - \boldsymbol{x}_1^*)^T(\boldsymbol{x}_1 - \boldsymbol{x}_1^*) + \frac{1}{2}(\boldsymbol{z}_1 - \boldsymbol{z}_1^*)^T(\boldsymbol{z}_1 - \boldsymbol{z}_1^*) \\ & + \frac{1}{2}(\boldsymbol{x}_2 - \boldsymbol{x}_2^*)^T(\boldsymbol{x}_2 - \boldsymbol{x}_2^*) + \frac{1}{2}(\boldsymbol{z}_2 - \boldsymbol{z}_2^*)^T(\boldsymbol{z}_2 - \boldsymbol{z}_2^*) \end{aligned}$$

Proof summary and consequences

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- convexity analysis (first-order condition of convexity, interplay F_1 and F_2)
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Consequences:

- analysis is also valid for **dynamic network connected topologies** (common Lyapunov function for switched system)
- convergence result valid also for ‘connected at times’ dynamic case

Robustness to link failures

What if agent interactions fail from time to time?

E.g., i receives information from j, but j does not receive it from i

Interaction topology becomes **directed**

- Different problem depending on **nature** and **frequency** of failures
- 'Closest' to undirected case is **weight-balanced digraph** (sum of weights of in-edges equals sum of weights of out-edges at each vertex)

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Mark also appreciates challenges posed by unidirectional information flows and nice structure behind weight-balanced digraphs

D. Lee and M. W. Spong. Stable flocking of multiple inertial agents on balanced graphs. IEEE Transactions on Automatic Control, *52(8):1469–1475, 2007*

Graph-theoretic notions – cont'd

Network topology modeled via **directed graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

- $\mathcal{V}$ is set of agent identities
- $\mathcal{E}$ is set of edges between agents – information sharing

Relevant **matrices** and their properties

- $\mathcal{A}$ is adjacency matrix (who interacts with whom)
- $L = \mathrm{diag}(\mathcal{A}\mathbf{1}_n) - \mathcal{A}$ is out-Laplacian matrix
- $L\mathbf{1}_n = 0$ (0 is an eigenvalue of L)
- $\mathcal{G}$ is strongly connected if and only if $\mathrm{rank}(L(\mathcal{G})) = n - 1$
- $\mathcal{G}$ is weight-balanced iff $\mathbf{1}_n^T L = 0$ iff $L + L^T$ positive semidefinite

Algorithm does not converge on digraphs

Algorithm in directed case 'looks' the same

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1 &= -\mathbf{L}_1 \boldsymbol{x}_1 - \mathbf{L}_1 \boldsymbol{z}_1 + \nabla_{\boldsymbol{x}_1} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\dot{\boldsymbol{z}}_1 &= \mathbf{L}_1 \boldsymbol{x}_1 \\
\dot{\boldsymbol{x}}_2 &= -\mathbf{L}_2 \boldsymbol{x}_2 - \mathbf{L}_2 \boldsymbol{z}_2 - \nabla_{\boldsymbol{x}_2} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\dot{\boldsymbol{z}}_2 &= \mathbf{L}_2 \boldsymbol{x}_2
\end{aligned}
$$

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\dot{\boldsymbol{z}}_2 &= \mathbf{L}_2 \boldsymbol{x}_2
\end{aligned}$$

but

- is no longer saddle dynamics (∇F_1, ∇F_2 have terms with $\mathbf{L}_a$ & $\mathbf{L}_a^T$)
- has correct equilibria only if graphs are weight-balanced

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but

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- has correct equilibria only if graphs are weight-balanced

Even worse, one can show that in general **dynamics is not convergent**!

- counterexample available
- surprising given what we know about weight-balanced digraphs

Distributed solution to directed 2-network 0-sum game

$$\begin{aligned}
\dot{\boldsymbol{x}}_1 &= -\alpha \mathbf{L}_1 \boldsymbol{x}_1 - \mathbf{L}_1 \boldsymbol{z}_1 + \nabla_{\boldsymbol{x}_1} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\dot{\boldsymbol{z}}_1 &= \mathbf{L}_1 \boldsymbol{x}_1 \\
\dot{\boldsymbol{x}}_2 &= -\alpha \mathbf{L}_2 \boldsymbol{x}_2 - \mathbf{L}_2 \boldsymbol{z}_2 - \nabla_{\boldsymbol{x}_2} \tilde{U}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
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\dot{\boldsymbol{z}}_2 &= \mathbf{L}_2 \boldsymbol{x}_2
\end{aligned}$$

Theorem

For zero-sum game, with Σ_1, Σ_2 strongly connected, weight-balanced, $U : \mathsf{X}_1 \times \mathsf{X}_2 \to \mathbb{R}$ strictly concave-convex and differentiable with globally Lipschitz gradient that admits lift to $\tilde{U}$, there is α_* such that for $\alpha \in (\alpha_*, \infty)$,

> *projection onto the first and third components of trajectories asymptotically converge to the Nash equilibrium*

[Specifically, $\alpha_* = \beta_* + 2/\beta_*$, where $\beta_* > 0$ is root of

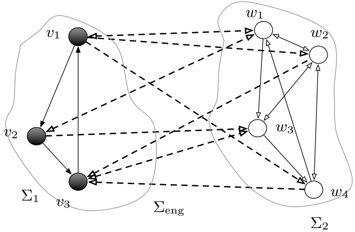
$$h(r) = \frac{1}{2} \Lambda_*^{\min} \Big(\sqrt{\Big(\frac{r^4 + 3r^2 + 2}{r}\Big)^2 - 4} - \frac{r^4 + 3r^2 + 2}{r} \Big) + \frac{K r^2}{(1 + r^2)}$$

$\Lambda_*^{\min} = \min_{a=1,2}\{\Lambda_*(\mathsf{L}_a + \mathsf{L}_a^T)\}$, $\Lambda_*(\cdot)$ smallest non-zero eigenvalue and K is Lipschitz constant of $\nabla \tilde{U}$]

Proof summary

Similar tools as undirected case – technically more challenging because of **unidirectional** interactions

- Lyapunov function of undirected case does not work
- Alternative function via understanding of counterexample
- Convexity analysis uses (novel) generalization of cocoercivity of concave-convex functions

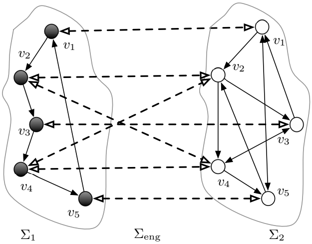


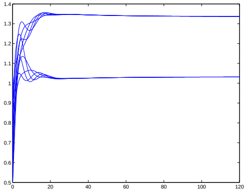
Simulation of scenario with communication channels

5 channels

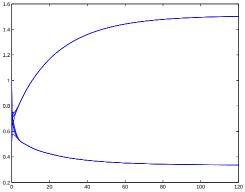
Σ_1 selects `ch1`, `ch3` with signal power x_1, `ch2`, `ch4` with signal power x_2

Σ_2 selects `ch1` with noise power y_1, `ch2`, `ch3`, `ch4` with noise power y_2

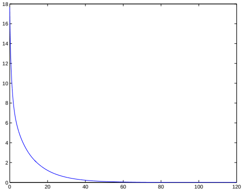




$\boldsymbol{x}(t)$



$\boldsymbol{y}(t)$



Lyapunov function

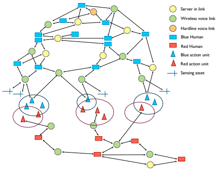
Summary

Conclusions

- strategic scenarios with partial information and distributed interactions
- distributed algorithms that converge to Nash equilibria
- dynamic interaction topologies, robustness to link failures

Future work

- robustness to noise
- interplay between strategic-distributed, transmission-acquisition of information
- hierarchy of layers with cooperation and competition
- deception mechanisms, robustness against deception



Happy Birthday Mark!